AIAA JOURNAL
Vol. 36, No. 8, August 1998

Three-Dimensional Free Vibration of Pretwisted Beams

Abdullah S. Balhaddad* and Dipo Onipede Jr.†
University of Pittsburgh, Pittsburgh, Pennsylvania 15261

A method for analyzing the free vibration of pretwisted structural members of arbitrary but uniform cross section is presented. This method is based on three-dimensional elasticity theory and three-dimensional finite element formulation. The equations of elasticity and the governing equations of motion are obtained using a rotating coordinate system that rotates along with the cross section. The relationship between the fixed Cartesian coordinate system and the rotating coordinate system is based on the pretwist angle rate. This method can capture warping, Poisson effects, and shear effects. Hamilton's principle is used to derive the equations of motion that, in turn, are used to obtain the natural frequencies and mode shapes. Results showing the relationship between the rate of pretwist and the natural frequencies of prismatic members with rectangular cross sections and different lengths are presented. The results also show coupling between the two bending modes and coupling between the axial and torsional modes. If the axis of rotation and the centroid of the cross section are not coincident, coupling among all four basic modes and higher modes may occur. This behavior is difficult or sometimes impossible to capture with simpler technical theories. Because the method is based on three-dimensional elasticity theory it can accurately capture the behavior of long or short beams, thin or thick sections, arbitrary geometry, and inhomogeneous, laminated (anisotropic) materials.

Nomenclature

$[C]$	= constitutive matrix of material properties
$[K]$	= stiffness matrix of beam
KE	= kinetic energy of system
L	= length of beam
$[M]$	= mass matrix of beam
$N(\xi, \eta, \zeta)$	= quadratic finite element interpolation functions
$[N]$	= matrix of interpolation functions
SE	= strain energy of system
U, V, W	= nodal displacements in rotating coordinate system
$\{\boldsymbol{u}\}$	= vector of displacements in rotating coordinate system
$\{\dot{\boldsymbol{u}}\}$	= vector of velocities
u, v, w	= displacements in the rotating coordinate system
u_x, u_y, u_z	= displacements in the fixed coordinate system
x, y, z	= fixed Cartesian coordinate system
α	= unit angle of pretwist
$\gamma_{\eta\zeta}\gamma_{\xi\zeta}\gamma_{\xi\eta}$	= shear strains in rotated coordinate system
$\varepsilon_{\xi\xi}\varepsilon_{\eta\eta}\varepsilon_{\zeta\zeta}$	= normal strains in rotated coordinate system
$\{\varepsilon\}$	= strain vector in rotated coordinate system
θ	= total angle of pretwist
ξ, η, ζ	= rotating coordinate system
ρ	= mass density
$\{\sigma\}$	= stress vector in rotated coordinate system
ω	= natural frequencies of system

Introduction

VIBRATION of pretwisted slender members has attracted many engineering designers and researchers because of their widespread applications. For example, blades of propellers, fans, helicopters, and turbines can be modeled as pretwisted beams. Curved beams, drills and end mills, and some gear teeth also can be modeled as pretwisted members. Improvement of aerodynamic and structural performance can be achieved by optimizing material layout, twist rate, and cross-sectional shapes of the pretwisted beam. A pretwisted beam is a beam whose cross section rotates about the perpendicular axis (z axis), and whose pretwist angle varies along the z axis.

Received Dec. 24, 1997; revision received May 4, 1998; accepted for publication May 7, 1998.

*Graduate Student, Mechanical Engineering Department; currently Vibration Engineer, Saudi Aramco, Dhahran 31311, Saudi Arabia.

†Assistant Professor, Mechanical Engineering Department, Benedum Engineering Hall. Member AIAA.

Figure 1 shows a pretwisted slender member. The global coordinates are fixed in space and the rotating coordinate system rotates with the cross section along the z axis. It is assumed that the pretwisted member is initially stress-free. The pretwist affects the natural frequencies, mode shapes, torsion and bending, and extensional stiffness of the beam. The pretwist also causes the coupling of the extensional, torsional, and bending modes. If the pretwist axis and the centroid of the cross section coincide, then coupling of the bending modes and extensional–torsional coupling is distinct.

The literature on pretwisted structural members is quite extensive. Rosen[1] conducted a comprehensive review on pretwisted rods and beams. His review addressed the static, dynamic, and stability aspects of pretwisted rods. Most of the works that he reviewed were based on one-dimensional models. Some two-dimensional, plate-and-shell theories and three-dimensional theories were covered. From Rosen's review, the status of three-dimensional elasticity studies can be summarized as very limited with most results devoted to isotropic members and restricted to elastic static analysis. In contrast, the body of literature on rod (one-dimensional) theories and analysis for both elastic–static and natural vibrations is ample.

One of the earliest researchers to investigate the effect of pretwist on rods was Love.[2] Love developed some equations that can give insight into the coupling phenomena found in rods with curved and twisted rods and beams.

Kosmatka[3] developed an analytical model of a pretwisted isotropic elastic beam with irregular cross section to study the coupled extension–bending–torsion coupling stiffnesses. Kosmatka[4] later extended this to composite pretwisted beams.

Onipede and Dong[5] developed a method to study end modes and propagating waves in pretwisted beams using three-dimensional elasticity theory and a semianalytical finite element formulation. They showed that the pretwisting causes a coupling of the extension, torsion, and flexural modes.

More recently, Lin[6] presented a method showing the coupled bending vibratory behavior of pretwisted nonuniform beams that are elastically restrained. His model was based on beam theory. Several other researchers have developed coupled one-dimensional beam-bar models in an attempt to capture the effect of pretwist.[7–11] Apart from that of Dong and Onipede, most of the dynamic models and theories presented were based on one-or-two-dimension theories.

The benefit of using a model based on three-dimensional elasticity to model the vibratory behavior of pretwisted beams is that it allows for inclusion of several types of structural behaviors that may not be possible using simpler technical theories. These include warping of

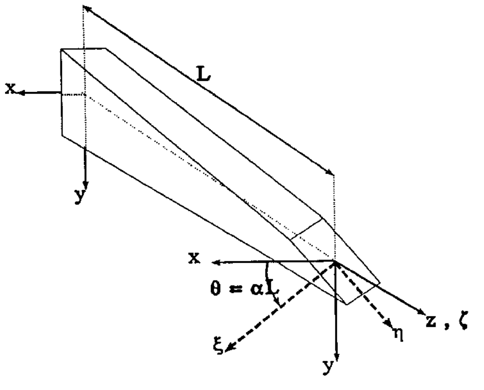

Fig. 1 Fixed and rotating coordinates of a pretwisted slender member.

the cross section, Poisson's ratio effects, high-frequency response, short-wavelength response, and shear effects.

Theoretical Background

Consider that a beam with arbitrary cross section rotates along the normal axis with constant rate. The geometry of the beam can be described best by using a rotating coordinate system. Examples shown are based on the coincidence between the centroid of the cross section and twist axis. The relationship between the fixed coordinate system and the rotating coordinate system is

$$\begin{Bmatrix} \xi \\ \eta \\ \zeta \end{Bmatrix} = \begin{bmatrix} \cos(\alpha z) & \sin(\alpha z) & \cdot \\ -\sin(\alpha z) & \cos(\alpha z) & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{1}$$

and α is constant over the length of the member. Figure 1 shows the relationship between the fixed and rotating coordinates.

The relationship between the displacements in the fixed coordinate system and in the rotating coordinate system also is based on the same transformation shown in Eq. (1). The strain–displacement relationship is expressed next (for a mathematical derivation of these expressions, see Ref. 12):

$$\begin{aligned} \varepsilon_{\xi\xi} &= \frac{\partial u}{\partial \xi}, & \gamma_{\eta\zeta} &= 2\varepsilon_{\eta\zeta} = \frac{\partial v}{\partial \zeta} + \alpha[u + D(v)] \\ \varepsilon_{\eta\eta} &= \frac{\partial v}{\partial \eta}, & \gamma_{\xi\zeta} &= 2\varepsilon_{\xi\zeta} = \frac{\partial u}{\partial \zeta} - \alpha[v - D(u)] \\ \varepsilon_{\zeta\zeta} &= \frac{\partial w}{\partial \zeta} + \alpha D(w), & \gamma_{\xi\eta} &= 2\varepsilon_{\xi\eta} = \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} \end{aligned} \tag{2}$$

where the operator $D(\,)$ is

$$D(\,) = \eta\frac{\partial(\,)}{\partial \xi} - \xi\frac{\partial(\,)}{\partial \eta} \tag{3}$$

The constitutive equations of an isotropic elastic material are of the form

$$\{\sigma\} = [C]\{\varepsilon\} \tag{4}$$

The $[C]$ matrix contains the constitutive equations for an elastic isotropic material, which are based on Poisson's ratio and modulus of elasticity.

Hamilton's Principle

The equations of motions are formulated by using Hamilton's principle. The total strain energy of the system can be expressed as

$$\mathrm{SE} = \frac{1}{2}\int_0^L \iint_\pi \{\varepsilon\}^T [C]\{\varepsilon\}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \tag{5}$$

The total kinetic energy is

$$\mathrm{KE} = \frac{1}{2}\int_0^L \iint_\pi \rho\{\dot{u}\}^T\{\dot{u}\}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \tag{6}$$

and the mass density ρ, is constant along the beam. Hamilton's variational principle leads to

$$\delta\int_{t_0}^{t_1} (\mathrm{KE} - \mathrm{SE})\,\mathrm{d}t = 0 \tag{7}$$

Finite Element Formulation

The cross section of the member is discretized into two-dimensional elements that are used to generate the third dimension along the length of the member. The cross section can be of any arbitrary shape having inhomogeneous, isotropic mechanical properties with the pretwist axis located at any point. Displacements of the discretized beam can be represented by

$$\begin{Bmatrix} u(\xi,\eta,\zeta,t) \\ v(\xi,\eta,\zeta,t) \\ w(\xi,\eta,\zeta,t) \end{Bmatrix} = \begin{bmatrix} N(\xi,\eta,\zeta) & \cdot & \cdot \\ \cdot & N(\xi,\eta,\zeta) & \cdot \\ \cdot & \cdot & N(\xi,\eta,\zeta) \end{bmatrix} \times \begin{Bmatrix} U(t) \\ V(t) \\ W(t) \end{Bmatrix} \tag{8}$$

Isoparametric 20-node brick finite elements with midside nodes are used with the computer program developed for this analysis. The brick element allows for the representation of any size and shape of quadrilateral region resulting from discretization of the cross section. Also, the brick's quadratic interpolation functions allow for a more accurate capturing of the curves of bending, torsion, and extension.[13,14]

Strain Transformation Matrices

The strain-displacement relationships shown in Eq. (2) can be expressed as a set of displacements and an operator matrix:

$$\begin{Bmatrix} \varepsilon_{\xi\xi} \\ \varepsilon_{\eta\eta} \\ \varepsilon_{\zeta\zeta} \\ \varepsilon_{\eta\zeta} \\ \varepsilon_{\xi\zeta} \\ \varepsilon_{\xi\eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial .}{\partial \xi} & \cdot & \cdot \\ \cdot & \frac{\partial .}{\partial \eta} & \cdot \\ \cdot & \cdot & \alpha D(.) + \frac{\partial .}{\partial \zeta} \\ \alpha & \alpha D(.) + \frac{\partial .}{\partial \zeta} & \frac{\partial .}{\partial \eta} \\ \alpha D(.) + \frac{\partial .}{\partial \zeta} & -\alpha. & \frac{\partial .}{\partial \xi} \\ \frac{\partial .}{\partial \eta} & \frac{\partial .}{\partial \xi} & \cdot \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{9}$$

Equation (8) now can be substituted into Eq. (9) and operator matrix allowed to operate on the interpolation functions as follows:

$$[\boldsymbol{B}] = \begin{bmatrix} \frac{\partial N}{\partial \xi} & \cdot & \cdot \\ \cdot & \frac{\partial N}{\partial \eta} & \cdot \\ \cdot & \cdot & \alpha D(N) + \frac{\partial N}{\partial \zeta} \\ \alpha N & \alpha D(N) + \frac{\partial N}{\partial \zeta} & \frac{\partial N}{\partial \eta} \\ \alpha D(N) + \frac{\partial N}{\partial \zeta} & -\alpha N & \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} & \frac{\partial N}{\partial \xi} & \cdot \end{bmatrix} \tag{10}$$

Governing Equation of Motion

Because Hamilton's principle is the basis for the finite element formulation for dynamic structural analysis, we can rewrite Eqs. (6) and (7) by using Eqs. (9) and (10):

$$\mathrm{KE} = \tfrac{1}{2}\{\dot{U}\}^T[M]\{\dot{U}\}, \qquad \mathrm{SE} = \tfrac{1}{2}\{U\}^T[K]\{U\} \tag{11}$$

where

$$[M] = \rho \int_{\mathrm{Vol}} [N]^T[N]\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

and (12)

$$[K] = \int_{\mathrm{Vol}} [B]^T[C][B]\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

Putting Eqs. (11) and (12) into Eq. (7), taking the variation, and assuming harmonic motion leads to the following:

$$[M]\{\ddot{u}\} + [K]\{u\} = 0, \qquad ([M]\omega^2 - [K])\{U\} = 0 \tag{13}$$

This system can be solved for the natural frequencies and mode shapes (nodal displacements) using a numerical procedure. (A computer program using Microsoft Fortran Power Station 4.0 with IMSL routines was developed for this analysis.)

Analysis of a Rectangular Beam

The method developed now is applied to the free vibration of isotropic rectangular beams with and without pretwist. The beam's cross-sectional dimensions are 1 unit × 1.5 units, and three different lengths of 5, 10, and 20 units are analyzed; these lengths represent short, moderate, and long beams, respectively. Poisson's ratio, Young's modulus, and mass density are 0.25, 2.6×10^9, and 1.0, respectively. The beam was modeled with nine elements in the cross section and five elements in the long direction, as shown in Fig. 2. Results from two cases of 0- and 40-deg pretwist are presented next.

Natural Frequencies

Figures 3–5 show the change in the natural frequencies as the total pretwist angle changes. The descriptors used in the graphs, i.e., weak bending (WB), strong bending (SB), torsional (T), and extensional (E), refer to the mode shape of the beam when no pretwist is present.

From these graphs some interesting but not surprising observations can be made. First, in all three cases the natural frequencies of the T and E modes are not affected significantly by the introduction of pretwist. Second, the SB mode initially decreases slightly then levels off or increases slightly. Third, the frequencies of the WB mode in most cases increase significantly.

Mode Shapes

Figures 6–13 show the cross-sectional mode shapes and are the output from the computer program developed for this analysis. The first line in each figure shows the in-plane deformations at the front and rear of the beam. The second line shows the out-of-plane deformations at these same locations. Separating the mode shapes this way allows for a clearer visual representation of the behavior of

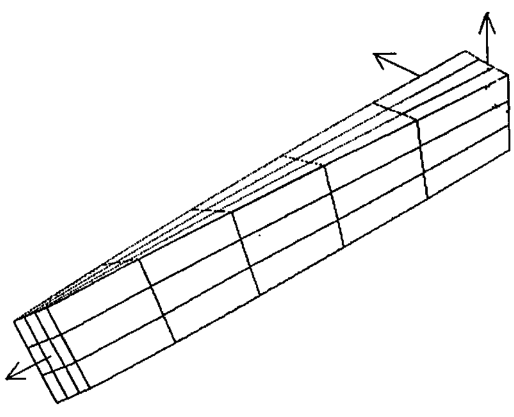

Fig. 2 Finite element model of pretwisted beam.

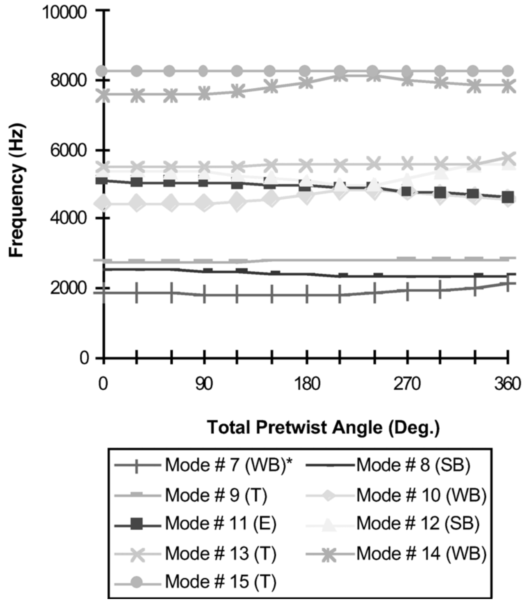

Fig. 3 Natural frequencies of a 5-unit-long beam.

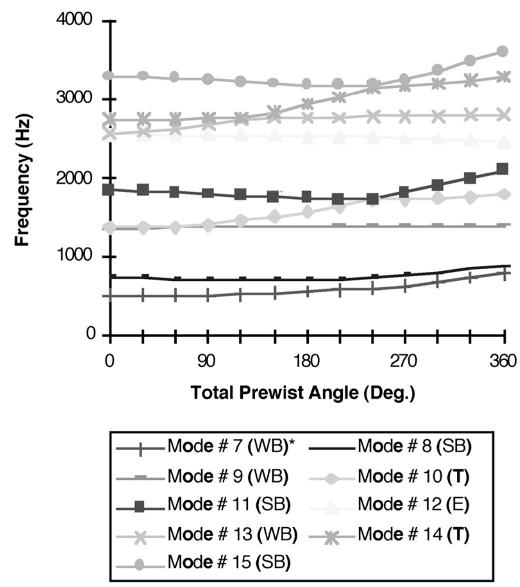

Fig. 4 Natural frequencies of a 10-unit-long beam.

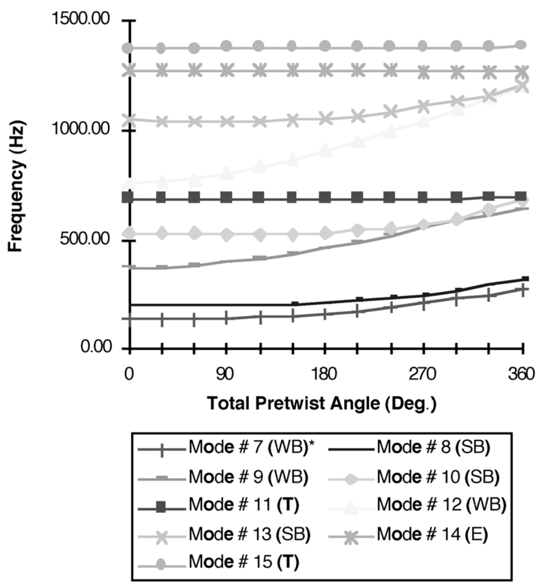

Fig. 5 Natural frequencies of a 20-unit-long beam.

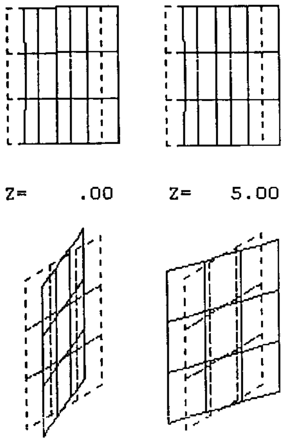

Fig. 6 Mode 7, weak bending mode ($\alpha = 0$ deg/unit length, $L = 5$ units).

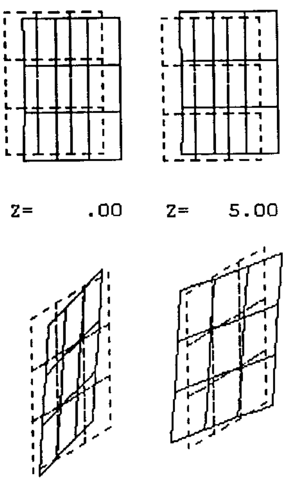

Fig. 7 Mode 7, weak bending mode ($\alpha = 8$ deg/unit length, $L = 5$ units).

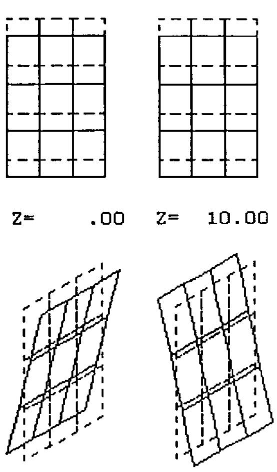

Fig. 8 Mode 8, strong bending mode ($\alpha = 0$ deg/unit length, $L =$ 10 units).

the beam. Only some of the mode shapes are shown because space considerations but these represent results from short, moderate, and long beams.

Verification of Results

To verify the results obtained from this method and the computer programs developed for this analysis, a commercial finite element program, ANSYS, was utilized. Twenty-node isoparametric brick elements were used to model straight beams of three lengths with and without pretwist. For the pretwist case, a total pretwist of 40 deg was used.

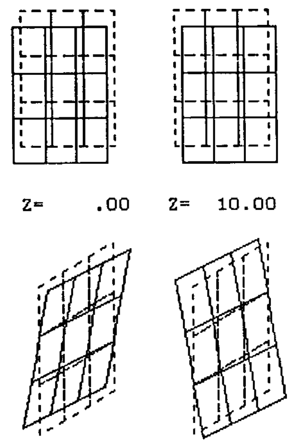

Fig. 9 Mode 8, strong bending mode ($\alpha = 4$ deg/unit length, $L =$ 10 units).

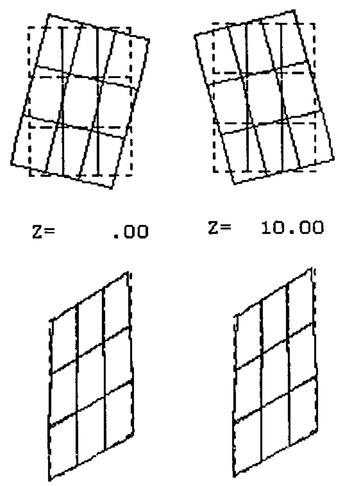

Fig. 10 Mode 10, torsional mode ($\alpha = 0$ deg/unit length, $L = 10$ units).

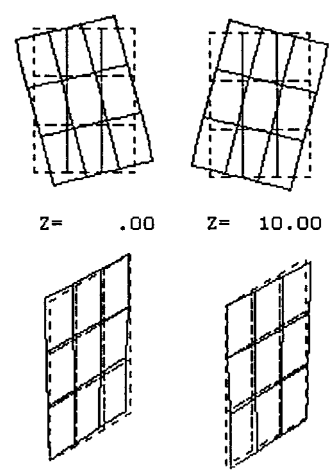

Fig. 11 Mode 10, torsional mode ($\alpha = 4$ deg/unit length, $L = 10$ units).

Table 1 compares the results for a 10-unit-long beam from ANSYS with those from the analysis presented in this paper. The results for the case of no pretwist are virtually identical. For the case of 40-deg total pretwist, there is a slight difference in the frequencies.

Similar results were obtained from comparisons undertaken for the case of 5- and 20-unit-long beams.[15] The differences in frequencies for the case of pretwist may be attributed to a number of possibilities. Modeling of the pretwist bar in ANSYS is based on an isoparametric formulation that distorts a 20-node brick (rectangular solid) element. This distortion may introduce errors into the results. It is also possible that in the formulation used in this analysis there is a limit on the size of the pretwist angle rate α.

Table 1 Natural frequencies of 10-unit-long rectangular beam

	No pretwist, Hz		40-deg total pretwist, Hz	
Mode[a]	ANSYS	Program	ANSYS	Program
7 (B)	509.9	509.9	510.3	511.6
8 (B)	734.5	734.5	730.2	732.1
9 (B)	1355.2	1355.2	1365.2	1367.0
10 (T)	1374.3	1374.2	1374.9	1374.6
11 (B)	1843.3	1843.3	1826.5	1833.4
12 (E)	2547.6	2547.6	2546.7	2546.7
13 (B)	2572.9	2572.8	2595.8	2595.4
14 (T)	2756.0	2754.8	2758.1	2755.6
15 (B)	3299.4	3299.4	3274.6	3289.4

[a]B, bending; T, torsion; E, extension.

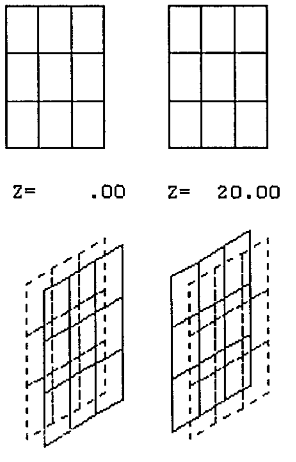

Fig. 12 Mode 14, extensional mode (α = 0 deg/unit length, L = 20 units).

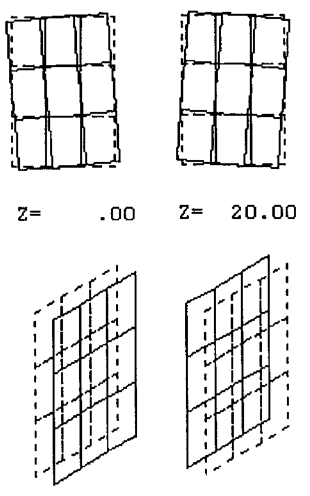

Fig. 13 Mode 14, extensional mode (α = 2 deg/unit length, L = 20 units).

Discussion of Results and Conclusions

From the results presented here and in related work, it is observed that the torsional and extensional frequencies of beams are not affected significantly by the introduction of pretwist. Mode shapes, on the other hand, may be affected more severely. The mode shapes show coupling between the extensional and torsional modes.

Both the frequencies and the mode shapes of the bending modes are affected significantly. In general, the frequency increases as either the total pretwist angle increases or the length of the beam increases. The mode shapes show coupling of the bending modes.

The coupling of mode shapes may indicate a significant change in the stress distribution of beams with pretwist even when the frequencies may not show much of a change. Simple technical theories may not be able to capture some of this structural behavior, especially the behavior and coupling found at higher frequencies.

References

[1]Rosen, A., "Structural and Dynamic Behavior of Pre-Twisted Rods and Beams," *Applied Mechanics Reviews*, Vol. 44, Dec. 1991, pp. 483–515.

[2]Love, A. E. H., "Small Deformation of Naturally Curved Rods," *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., Dover, New York, 1927, pp . 444–447.

[3]Kosmatka, J. B., "On the Behavior of Pre-Twisted Beams with Irregular Cross-Sections," *Journal of Applied Mechanics*, Vol. 59, March 1992, pp. 146–152.

[4]Kosmatka, J. B., "Extension-Bend-Twist Coupling Behavior of Nonhomogeneous Anisotropic Beams with Initial Twist," *AIAA Journal*, Vol. 30, No. 2, 1992, pp. 519–527.

[5]Onipede, O., and Dong, S. B., "Propagating Waves and End Modes in Pre-Twisted Beams," *Journal of Sound and Vibration*, Vol. 195, No. 2, 1996, pp. 313–330.

[6]Lin, S. M., "Vibrations of Elastically Restrained Nonuniform Beams with Arbitrary Pretwist," *AIAA Journal*, Vol. 35, No 11, 1997, pp. 1681–1687.

[7]Dawson, B., "Coupled Bending-Bending Vibrations of Pre-Twisted Cantilever Blading Treated by the Rayleigh-Ritz Energy Method," *Journal of Mechanical Engineering Science*, Vol. 10, No. 5, 1968, pp. 381–388.

[8]Gupta, R. S., and Rao, S. S., "Finite Element Eigenvalue Analysis of Tapered and Twisted Timoshenko Beams," *Journal of Sound and Vibration*, Vol. 56, 1978, pp. 187–200.

[9]Sabuncu, M., "Coupled Vibration Analysis of Blades with Angular Pre-Twist of Cubic Distribution," *AIAA Journal*, Vol. 23, No. 9, 1985, pp. 1424–1430.

[10]Slyper, H. A., "Coupled Bending Vibrations of Pre-Twisted Cantilever Beams," *Journal of Mechanical Engineering Science*, Vol. 4, No. 4, 1962, pp. 365–379.

[11]Kaza, K. R. V., and Kielb, R. E., "Effects of Warping and Pre-Twist on Torsional Vibration of Rotating Beams," *Journal of Applied Mechanics*, Vol. 51, No. 4, 1984, pp. 913–919.

[12]Onipede, O., "Vibration and Wave Propagation in Pre-Twisted Beams and Rods," Ph.D. Dissertation, Dept. of Civil Engineering, School of Engineering, Univ. of California, Los Angeles, CA, 1995, pp. 114–120.

[13]Logan, D. L., *A First Course in the Finite Element Method*, PWS, Boston, MA, 1993, pp. 427–431.

[14]Zienkiewicz, O. C., and Taylor, R. L., "Rectangular Prisms—Serendipity Family," *The Finite Element Method*, 4th ed., Vol. 1, McGraw–Hill (UK), Maidenhead, England, UK, 1977, pp. 133, 134.

[15]Balhaddad, A., "3-Dimensional Free Vibration of Pre-Twisted Beams," M.S. Thesis, Dept. of Mechanical Engineering, School of Engineering, Univ. of Pittsburgh, Pittsburgh, PA, 1997, pp. 24–57.

A. Berman
Associate Editor